PMM U.S.S.R.,Vol.50,No.1,pp.126-129,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

CONSTRUCTION OF SIMPLIFIED EQUATIONS OF NON-LINEAR DYNAMICS OF PLATES AND SHALLOW SHELLS BY THE AVERAGING METHOD*

I.V. ANDRIANOV

A sequential procedure for deriving of equations of the Berger equation (BE) type is described that relies upon the method of averaging /1—3/ for rectangular and circular isotropic plates and isotropic and sandwich shallow shells. It is shown that the smallness of the second invariant of the strain tensor is random in nature. The Berger hypothesis (BH) /4/ holds in pure form only for isotropic single-layer and transversely-isotropic sandwich plates, while the averaging idea is necessary for the sequential construction of a simplified Berger theory.

1. We first present several intuitive considerations (their usefulness is demonstrated graphically in /5/ where reasoning is presented about the application of the averaging method in the case of rapid variability in non-linear vibrational systems). The validity of the BH for isotropic rectangular plates is confirmed by a large number of computations /4—6,7/ and raises no doubts that the contribution of the second invariant I_2 of the strain tensor to the potential energy is substantially less than the contribution of the first invariant I_1. Taking into account that

$$I_1 = \varepsilon_1 + \varepsilon_2,\ I_2 = \varepsilon_1\varepsilon_2 - {}^1/_4\varepsilon_{12}^2$$
$$\varepsilon_1 = u_x + {}^1/_2 w_x^2,\ \varepsilon_2 = v_y + {}^1/_2 w_y^2$$
$$\varepsilon_{12} = u_y + v_x + w_x w_y$$

the appropriate inequality for a rectangular plate $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b)$ can be rewritten as

$$\int_0^b\int_0^a (A + B_1 + C)\,dx\,dy \ll (1-\nu)\int_0^b\int_0^a (A - B_2)\,dx\,dy \tag{1.1}$$
$$A = 2u_x y_y + w_y^2 u_x + w_x^2 v_y,\ B_1 = B_{11} + B_{12},\ B_2 = {}^1/_2 B_{11} + B_{22}$$
$$B_{11} = u_x^2 + u_y^2,\ B_{12} = u_x w_x^2 + v_y w_y^2$$
$$B_{22} = (u_y + v_x)\,w_x w_y,\ C = {}^1/_4\,(w_x^2 + w_y^2)^2$$

The fundamental difference between the left and right sides of inequality (1.1) is related to the component C. Indeed, let the displacements and bending moments be zero on the plate edges and let the problem of natural vibrations be considered. Applying the Bubnov-Galerkin method in the first approximation

$$(u, v, w) = A_i\,(t)\sin(m\pi x/a)\sin(n\pi y/b)$$

we see that the contribution of the expressions $(A + B_1)$ and $(A - B_2)$ to the potential energy is approximately identical (in order of magnitude in every case), with the exception of the special case $a = b,\ m = n$. This means that the contribution of the component C to the potential energy must be predominant to satisfy the inequality (1.1). This will hold for $m \approx n \gg 1$. Then the component C is large in absolute value because of differentiation $(C \gg B_{11})$. Moreover, the term C contains a mean part, while the terms B_{11}, B_{22} have only a fast part; consequently, integrals of them are small. The reasoning presented suggests using the averaging method /1—3/ for BE construction for large variability in the spatial variables.

The idea of using the averaging method in the case under considerations was expressed earlier /8, 9/, but it had not been realized sequentially.

2. We will write the equations of motion of a rectangular plate in dimensionless form

$$[12\,(1-\nu^2)]^{-1}\varepsilon^2\nabla^2\nabla^2 w' + \varepsilon\,(F_{\xi\xi}' w_{\eta\eta}' - 2F_{\xi\eta}' w_{\xi\eta}' + F_{\eta\eta}' w_{\xi\xi}') + w_{\tau\tau}' = 0 \tag{2.1}$$
$$\nabla^2\nabla^2 F' + \varepsilon\,(w_{\xi\xi}' w_{\eta\eta}' - w_{\xi\eta}'^2) = 0$$
$$F_{\eta\eta}' = (1-\nu^2)^{-1}\,[u_\xi' + {}^1/_2\varepsilon w_\xi'^2 + \nu\,(v_\eta' + {}^1/_2 w_\eta'^2)]$$
$$F_{\xi\xi}' = (1-\nu^2)^{-1}\,[v_\eta' + {}^1/_2\varepsilon w_\eta'^2 + \nu\,(u_\xi' + {}^1/_2 w_\xi'^2]$$
$$F_{\xi\eta}' = -{}^1/_2\,(1+\nu)^{-1}\,(u_\eta' + v_\xi' + \varepsilon w_\xi' w_\eta')$$
$$\varepsilon = \frac{h}{a},\quad (\xi, \eta) = \left(\frac{x}{a}, \frac{y}{a}\right),\quad F' = \frac{F}{Eha}$$
$$(u', v', w') = \frac{(u, v, w)}{h},\quad \tau = at\sqrt{\frac{\rho\,(1-\nu^2)}{E}},\quad \nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}$$

*Prikl.Matem.Mekhan.,50,1,171-174,1986

The concept of rapid variability in a non-linear system is introduced most naturally by using a new fast variable $\varepsilon^\alpha\theta(\xi,\eta)$ $(\alpha<0)$, considered the independent variable. The specific value of α is determined during the construction of limit systems $(\varepsilon\to 0)$. Now, in conformity with the method of two scales /1—3/, we obtain (keeping the notation ξ,η) for the slow variables)

$$\frac{\partial}{\partial\xi}=\frac{\partial}{\partial\xi}+\varepsilon^\alpha\theta_\xi\frac{\partial}{\partial\theta},\quad \frac{\partial}{\partial\eta}=\frac{\partial}{\partial\eta}+\varepsilon^\alpha\theta_\eta\frac{\partial}{\partial\theta}$$

We represent the functions F', w', u', v' in the form of sums of slow (i.e., dependent only on the slow independent variables) and fast periodic components with the unknown period $\theta_0(\xi,\eta)$ /1—3/

$$F'=F^0+\varepsilon^{\beta_1}F^1,\quad w'=w^0+\varepsilon^{\beta_2}w^1,\quad u'=u^0+\varepsilon^{\beta_3}u^1,\quad v'=v^0+\varepsilon^{\beta_4}v^1$$

The quantities with superscript 0 here are functions of the variables ξ,η, while those with superscript 1 are functions of $\xi,\eta,\varepsilon^\alpha\theta$.

We also introduce the relationships

$$F^0\sim\varepsilon^{\gamma_1}w^0,\quad w^0\sim\varepsilon^{\gamma_2},\quad u^0\sim\varepsilon^{\gamma_3},\quad v^0\sim\varepsilon^{\gamma_4},\quad \frac{\partial}{\partial\tau}(\ldots)\sim\varepsilon^\delta(\ldots)$$

where $\beta_i, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta$ are asymptotic integration parameters.

Sampling of consistent values of these parameters is realized from the consistency conditions of the limit systems $(\varepsilon\to 0)$

A non-trivial limit system is obtained from the initial (2.1) for $\alpha=-{}^1/_2$, $\beta_1=0, \beta_2<0, \beta_3,$ $\beta_4\geqslant -{}^1/_2, \gamma_1=0, \gamma_2=0, \gamma_3,\gamma_4>0, \delta=0$ and has the form

$$[12(1-\nu^2)]^{-1}w^1_{\theta\theta\theta\theta}(\theta_\xi^2+\theta_\eta^2)^2+\Lambda w^1_{\theta\theta}+w^1_{\tau\tau}=0 \tag{2.2}$$

$$F^1_{\theta\theta\theta\theta}(\theta_\xi^2+\theta_\eta^2)=0 \tag{2.3}$$

$$\Lambda=F^0_{\xi\xi}\theta_\eta^2-2F^0_{\xi\eta}\theta_\xi\theta_\eta+F^0_{\eta\eta}\theta_\eta^2$$

$$\varepsilon^{-1}F^1_{\theta\theta}\theta_\eta^2+F^0_{\eta\eta}={}^1/_2(1-\nu^2)^{-1}(\omega_\theta^1)^2(\theta_\xi^2+\nu\theta_\eta^2)$$

$$\varepsilon^{-1}F^1_{\theta\theta}\theta_\xi^2+F^0_{\xi\xi}={}^1/_2(1-\nu^2)^{-1}(w_\theta^1)^2(0_\eta^2+\nu\theta_\xi^2)$$

$$\varepsilon^{-1}F^1_{\theta\theta}\theta_\xi\theta_\eta+F^0_{\xi\eta}=-{}^1/_2(1+\nu)^{-1}(w_\theta^1)^2\theta_\xi\theta_\eta$$

We note that the limit systems describing the dynamics and statics of linear and non-linear rods and a linear plate are of no interest in this case and are, consequently, not presented.

The term Λ in (2.2) can be determined by using the averaging operator in θ

$$\langle\cdot\rangle=\theta_0^{-1}\int_0^{\theta_0}(\cdot)\,d\theta$$

Here

$$\langle F_{\theta\theta}\rangle=0,\quad \langle F^0_{\xi\xi},F^0_{\eta\eta},F^0_{\xi\eta}\rangle=(F^0_{\xi\xi},F^0_{\eta\eta},F^0_{\xi\eta})$$

$$\langle\Lambda\rangle={}^1/_2(1-\nu^2)^{-1}\langle w^{Q1}\rangle^2(\theta_\xi^2+\theta_\eta^2)$$

In the initial variables

$$\langle w_\theta^1\rangle^2(\theta_\xi^2+\theta_\eta^2)=\frac{1}{ab}\int_0^b\int_0^a(w_x^2+w_y^2)\,dx\,dy+O(\varepsilon)$$

and (2.2) goes over into the BE

$$D\nabla^2\nabla^2w+N\nabla^2w+\rho hw_{tt}=0 \tag{2.4}$$

$$N=\frac{B}{2ab}\int_0^b\int_0^a(w_x^2+w_y^2)\,dx\,dy$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad D=\frac{Eh^2B}{12},\quad B=\frac{Eh}{1-\nu^2}$$

which is therefore obtained without using the BH. Unlike /10/, (2.4) is obtained directly from the von Karman equation.

The strain compatibility equation is linearized

$$\nabla^4F=0 \tag{2.5}$$

Saint-Venant /11/ even proposed a similar simplification in the analysis of non-linear plates; however, he kept the equilibrium equation in its previous form, which is inconsistent, as is seen from the computations presented.

The BE for a viscoelastic plate (R is the relaxation kernel) can be obtained as above

$$D\Gamma\nabla^2\nabla^2 w - \Gamma N\nabla^2 w + \rho h w_{tt} = 0, \quad \Gamma\varphi = \varphi + \int_0^{\tau_1} R(\tau - \tau_1)\varphi\, d\tau_1$$

For circular plates ($r_0 \leqslant r \leqslant R,\ 0 \leqslant \theta \leqslant 2\pi$) we obtain (2.4) where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \quad N = \frac{B}{2\pi R^2}\int_0^{2\pi}\int_0^R (w_r^2 + w_\theta^2)\, dr\, d\theta$$

3. Now we consider a shallow isotropic shell with curvatures k_1, k_2 and linear planform dimensions a, b. Considering $a \sim b,\ k_i \sim 1$, and performing computations analogous to the case of the plate, we arrive at equations of BE type for a shallow shell

$$D\nabla^4 w + h\nabla_k F + N\nabla^2 w + \frac{B}{ab}\left[w_{xx}\int_0^b\int_0^a (k_1 + \nu k_2)\, w\, dx\, dy + \right. \tag{3.1}$$

$$\left. w_{yy}\int_0^b\int_0^a (k_2 + \nu k_1)\, w\, dx\, dy\right] + \rho h w_{tt} = 0,\ \nabla^4 F + E\nabla_k w = 0, \qquad \nabla_k = k_1\frac{\partial^2}{\partial x^2} + k_2\frac{\partial^2}{\partial y^2}.$$

We emphasize that Eqs.(3.1) allow of all natural passages to the limit: to the Berger plate, to a non-linear Kirchhoff rod /12, 13/, to a shallow arch /12/, and finally (stumbling blocks of shallow shell equations based on the BH /6, 7, 14-20/), to linear shallow shell equations. We also note that the BH is not true for (3.1) (the energy of the second invariant of the strain tensor is not small compared with the energy of the first invariant).

We will obtain approximate equations for transversely-isotropic sandwich shells. The initial non-linear equations are presented in /21/ while the estimates of the parameters in them are in /22, 23/. We introduce the relationships

$$D_0 = D\theta_0, \qquad \mu = h^2 k_1 \beta^{-2}, \qquad \rho_1 = \sum_{k=1}^{3} \rho_k h_k$$

where θ and μ are parameters characterizing the bending stiffness of the carrying layers and the compliance to shear of the sandwich packet (formulas to compute θ, θ_0, β are presented in /21/), and ρ_k, h_k are the specific densities of the materials and the thickness of the k-th layer.

If the estimates $\theta_0 \sim 1,\ \mu \sim h/R,\ \theta \sim 1$ are taken, then the approximate equations take the form

$$D_0(1 - \{\theta\beta^{-1}R^2\nabla^2\})\nabla^2\nabla^2 X + h\nabla_k F + N\nabla^2 w + \tag{3.2}$$

$$\frac{B}{ab}\left[w_{xx}\int_0^b\int_0^a (k_1 + \nu k_2)\, w\, dx\, dy + w_{yy}\int_0^b\int_0^a (k_2 + \nu k_1)\, w\, dx\, dy\right] + \rho_1 w_{tt} = 0$$

$$\nabla^4 F + E\nabla_k w = 0 \tag{3.3}$$

$${}^1/_2(1 - \nu)\,\mu R^2\nabla^2\varphi = \varphi, \qquad w = (1 - \mu R^2\nabla^2)\, X \tag{3.4}$$

For plates ($k_1 = k_2 = 0$) Eqs.(3.2) and (3.4) agree with those obtained earlier in /24, 25/ on the basis of the BH. If $\theta < 1$, the term in the braces should be discarded in (3.2).

Thus, the BH in its original form is true only for isotropic single-layers and transversely isotropic laminar plates. The true meaning of the BE is that they are the first approximations of the averaging method for fast variability in the space variables. This provides the possibility of effective utilization of the BE to extend Bolotin's asymptotic method to the non-linear case /26/.

REFERENCES

1. LUKE J.C., A perturbation method for non-linear dispersive wave problems. Proc. Roy. Soc. Ser. A, 292, 1430, 1966.
2. WHITHAM G., Linear and Non-linear Waves, Mir, Moscow, /Russian translation/, 1977.
3. LEIBOVICH L.S. and SEABASS A., eds. Non-linear Waves, Mir, Moscow, /Russian translation/, 1977.
4. BERGER H.M., A new approach to the analysis of large deflections of plates. J. Appl. Mech. 22, 4, 1955.
5. BLEKHMAN I.K., MYSHKIS A.D. and PANOVKO YA.G., Applied Mathematics: Subject, Logic, Features of Approaches, Naukova Dumka, Kiev, 1976.
6. PRATHAR G., On the Berger approximation: a critical re-examination, J. Sound and Vibr., 66, 2, 1979.
7. GRIGOLYUK E.I. and KULIKOV G.M., On a simplified method of solving non-linear problems of the theory of elastic plates and shells. Certain Applied Problems of the Theory of Elastic Plates and Shells. Izdat. Moskov. Gosudarst. Univ., Moscow, 1981.

8. SIBUKAYEV SH.M., Vibrations of elastic plates under large deflections. Trudy Tashkent. Univ., 275, 1966.
9. KORNEV V.M., On a simplified model of non-linear shell theory. Dynamics of a Continuous Medium, 49, Izd, Inst. Gidrodinamiki Sibirsk. Otdel. Akad. Nauk SSSR, 1981.
10. ANDRIANOV I.V., On the theory of Berger plates, PMM, 47, 1, 1983.
11. GRIGOLYUK E.I. and KULIKOV G.M., Approximate analysis of anisotropic sandwich plates of finite deflection, Mekhan. Kompositn. Materialov, 1, 1980.
12. KAUDERER H., Non-linear Mechanics. Izd. Inostr. Lit., Moscow, /Russian translation/, 1961.
13. ANDRIANOV I.V. and MANEVICH L.I., Approximate equations of axisymmetric vibrations of cylindrical shells, Prikl. Mekhan., 17, 8, 1981.
14. JONES R., Remarks on the approximate analysis of the non-linear behaviour of shallow shells. J. Struct. Mech., 3, 2, 1975.
15. NOWINSKI J.L. and ISMAIL I.A., Certain approximate analyses of large deflections of cylindrical shells, Z. Angew. Math. und. Phys., 15, 15, 1964.
16. NASH W.A. and MODEER J.R., Certain approximate analyses of the non-linear behaviour of plates and shallow shells. Proc. Symp. on Theory of Thin Shells, Delft, 1959, North-Holland, Amsterdam, 1960.
17. ANNIN B.D. and KHLUDNEV A.M., Existence and uniqueness of the solution of non-linear rod and plate vibrations. Mechanics of Deformable Bodies and Structures. 39-43, Mashinostroenie, Moscow, 1975.
18. KHLUDNEV A.M., On an equation of shallow shell theory. Dinamika Sploshnoi sredy, 21, 84-98, Izd. Inst. Gidrodinamiki, Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1975.
19. RAMACHANDRAN J., Vibration of shallow spherical shells of large amplitudes, Trans. ASME, Ser. E., J. Appl. Mech., 41, 3, 1974.
20. BUCCO D., JONES R. and MASUMDAR J., The dynamic analysis of shallow spherical shells. Trans. ASME, J. Appl. Mech., 45, 3, 1978.
21. GRIGOLYUK E.I. and CHULKOV V.P., Stability and Vibrations of Sandwich Shells. Mashinostroenie, Moscow, 1973.
22. GRIGOLYUK E.I. and KORNEV V.M.: Asymptotic investigation of the equations of non-symmetric bending of a multilayered cylindrical shell. In: Theory of Plates and Shells, Moscow, Nauka, 1971.
23. VAKHROMEYEV YU. and KORNEV V.M., On the asymptotic analysis of the stress-deformation state of three-layered cylindrical shells. In: Dynamics of Continous Media, Novosibirsk, publishing house of the Institute of Hydrodynamics, Academy of Sciences of the USSR, 22, 1975.
24. ALEKSEYEVA N.K., Approximate method of determining the natural frequencies of flexible rectangular plates. "Applied Mechanics", 9, 6, 1973.
25. GRIGOLYUK E.I. and KULIKOV G.M., Approximate analysis of non-linear transversely isotropic three-layered plates. "Mechnics of Composite Materials", 2, 1980.
26. ANDRIANOV I.V., MANEVICH L.I. and KHOLOD E.G., On non-linear oscillations of rectangular plates. "Structural Mechanics and Analysis of Structures", 5, 1979.

Translated by M.D.F.